SOLITARY WAVES IN MAGMA DYNAMICS

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ABSTRACT. Movement of magma through the earth's mantle can be seen as porous media flow. To model this molten rock migration we derive a model describing a viscous fluid flow through a compacting solid matrix based on work by McKenzie [10]. Upon simplification using, inter alia, a Boussinesq approximation we arrive at a more tractable non-linear model with rich behaviour. Solitary wave type solutions have been observed numerically for simplified models [3, 13]. Analytic expressions for a restricted class of travelling waves are derived which confirm the existence of solitary waves. Strong evidence to date suggests that these special wave solutions are not solitons.

1. INTRODUCTION

The dynamics of the earth's crust and mantle have been subject of intense study by geophysicists. One of the problems encountered in the study of the outer layers of the earth is related to the presence of molten rock or magma. It is observed that the crust and mantle seem to contain more highly concentrated regions of magma, such as magma chambers, than can be expected from experiments looking at the partial melting of rocks. Note that one might intuitively consider this partial melting of rocks as the main source of magma. As a result it is believed that large-scale migration of molten rocks is an important factor that has to be incorporated in models of the earth's mantle. The combination of creation of melt in the outer layers and its transport attracted attention from the geophysical community in the eighties, when mathematical models of this mechanism were proposed. The majority of these models were based on the idea of a two-phase flow consisting of a melt and its solid phase. The most successful of these was proposed by McKenzie [10] in 1984, describing the compaction and formation of partially molten rock derived from conservation laws. Simultaneously equations similar in spirit were derived independently for the flow of water through glaciers by Fowler [6]. The set of equations in the context of magma dynamics are nowadays known under the name McKenzie equations.

After the introduction of the McKenzie model, its analysis showed interesting dynamics. One of the most prominent features is the apparent formation of solitons, called *magmons* in the literature. This was first observed by Scott & Stevenson [13] in 1984 in a simplified model leading to the so-called magma equation. Later analysis of these solitary wave-type solutions by Barcilon & Richter [3] showed evidence for the solutions not being pure solitons (as defined by Drazin & Johnson [4]).

This paper will follow the historical order of the development of solitary wave solutions in magma dynamics, by first considering a derivation of the governing equations, the McKenzie

equations and the closely related magma equation. Thereafter, we will look at the travelling solitary wave solutions and the question whether they are solitons or not.

2. Magma dynamics: McKenzie equations

We will consider the dynamics of a highly viscous fluid, the melt, in a porous matrix, following the successful introduction by McKenzie in 1984 [10]. The high viscosity is justified by the very low Reynolds' number, of the order 10^{-8} or smaller [10], observed for both the flow of the melt as that of the matrix. As a result we will ignore inertial effects in the model.

The model is inspired by observations of the local microscopic environment of the melt/matrix mixture. Based on observations (e.g. Zhu et al. [15]) we assume the magma makes up an interconnected network of small tubes in between solid grains of rock, see figure 1. The typical length scale of the grains $a \sim 10^{-3}$ m [10, 15] is assumed to be much smaller than the typical global scale at which the magma is moving ($\sim 10^3$ m) so that we can derive a macroscopic model for a two-phase fluid.
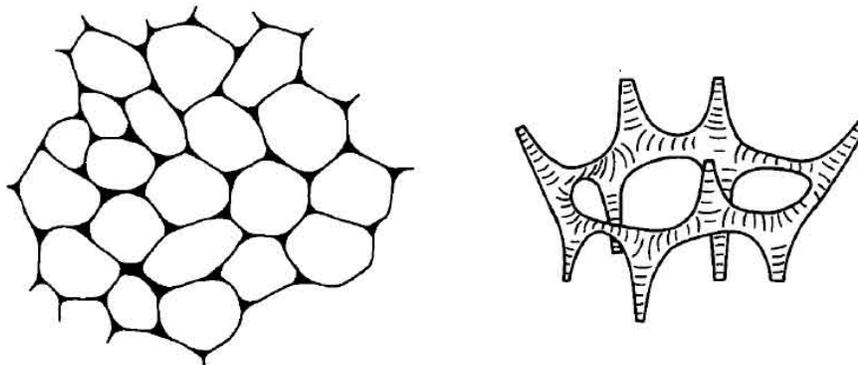
(A) Two-dimensional slice of grain and magma distribution.

(B) Three-dimensional interconnected melt tubes forming a network.

FIGURE 1. Sketch by Barcilon et al. [3] of the melt grain distribution on microscopic scale. The melt makes up a network of connected tubes surrounding the solid grains. Suggested by evidence we assume that tubes stay interconnected even for low volume fractions of magma [10].

To this effect, we consider a reference volume element $V(t)$, which is being convected by one of the fluids. The size of the volume element must be larger than the microscopic scale a, so as to allow averaging over the volume, neglecting individual magma pores. On the other hand it has to be much smaller than any macroscopic scale as we do not want to average out the global dynamics. To derive dynamic equations we use Reynolds' transport theorem, i.e. for any scalar function $f(\mathbf{x}, t)$

$$(1) \qquad \frac{d}{dt} \int_{V(t)} f \, dV = \int_{V(t)} \frac{\partial f}{\partial t} + \nabla \cdot (f\mathbf{v}) \, dV,$$

where $\mathbf{v}$ is the velocity of the convecting fluid under consideration. As we need to distinguish between melt and solid we will denote the different variables with subscripts m and s respectively. Lastly we introduce the porosity ϕ as a ratio of melt and solid volume [5]

$$(2) \qquad \phi = \frac{1}{V} \int_{V(t)} \mathbb{1} \, dV,$$

where $\mathbb{1}$ is an indicator function

$$(3) \qquad \mathbb{1}(\mathbf{x}) = \begin{cases} 1 & \text{if melt at } \mathbf{x}, \\ 0 & \text{if solid at } \mathbf{x}. \end{cases}$$

2.1. Conservation laws.

Conservation of mass. We start with the conservation of mass. The mass of both phases in a volume element is found by integrating $\phi\rho_m$ and $(1-\phi)\rho_s$ respectively for melt and solid. Conservation of mass then reads

$$(4) \qquad \int_{V(t)} \frac{\partial\phi\rho_m}{\partial t} + \nabla\cdot(\phi\rho_m\mathbf{v}_m)\,\mathrm{d}V = \frac{d}{dt}\int_{V(t)}\phi\rho_m\,\mathrm{d}V = \int_{V(t)}\Gamma\,\mathrm{d}V,$$

$$(5) \qquad \int_{V(t)} \frac{\partial(1-\phi)\rho_s}{\partial t} + \nabla\cdot((1-\phi)\rho_s\mathbf{v}_s)\,\mathrm{d}V = \frac{d}{dt}\int_{V(t)}(1-\phi)\rho_s\,\mathrm{d}V = \int_{V(t)}-\Gamma\,\mathrm{d}V,$$

where Γ is the mass-transfer rate per unit volume from solid phase to melt, i.e. the melting rate. As the volume is arbitrary we arrive at the continuum relations describing conservation of mass

$$(6) \qquad \frac{\partial\phi\rho_m}{\partial t} + \nabla\cdot(\phi\rho_m\mathbf{v}_m) = \Gamma,$$

$$(7) \qquad \frac{\partial(1-\phi)\rho_s}{\partial t} + \nabla\cdot((1-\phi)\rho_s\mathbf{v}_s) = -\Gamma.$$

Note that Γ is unspecified and needs to be prescribed by a constitutive law describing conservation of energy. See Spiegelman [14] for an example of the dynamics of mid-ocean ridges. In this paper we will only consider dynamics without melting, i.e. $\Gamma = 0$, following the early analysis of the McKenzie equations which used this assumption mainly for simplicity reasons [2, 3, 11–13]. In section 2.2 another motivation for neglecting melting and freezing will be given.

Conservation of momentum. In the conservation of momentum we must now include interactions between the melt and the surrounding matrix, the gravitational pull and Cauchy stresses. This yields that

$$(8) \qquad \frac{d}{dt}\int_{V(t)}\phi\rho_m\mathbf{v}_m\,\mathrm{d}V = \int_{V(t)}\phi\rho_m\mathbf{g}\,\mathrm{d}V - \int_{V(t)}\mathbf{f}_I\,\mathrm{d}V + \int_{V(t)}\nabla\cdot(\phi\boldsymbol{\sigma}_m)\,\mathrm{d}V,$$

$$(9) \qquad \frac{d}{dt}\int_{V(t)}(1-\phi)\rho_s\mathbf{v}_s\,\mathrm{d}V = \int_{V(t)}(1-\phi)\rho_s\mathbf{g}\,\mathrm{d}V + \int_{V(t)}\mathbf{f}_I\,\mathrm{d}V + \int_{V(t)}\nabla\cdot((1-\phi)\boldsymbol{\sigma}_s)\,\mathrm{d}V,$$

where $\mathbf{g}$ represents gravitational acceleration, $\mathbf{f}_I$ the melt-matrix interaction force per body volume produced by moving melt on the soid matrix and $\boldsymbol{\sigma}_{m,s}$ the Cauchy stress tensor for melt and solid. Because of Newton's third law the reaction force from the matrix on the moving melt is given by $-\mathbf{f}_I$. Note that the stress tensor is multiplied by the proper voidage factor. This can be understood by looking at the origin of the stress term, i.e. surface forces. The stresses only act along that part of the surface that is occupied by the right phase yielding a voidage factor in the surface integral deriving the total Cauchy stress on the volume. The Cauchy stress tensor is then derived using the divergence theorem, which puts the porosity inside the divergence.

Recall that we are considering low Reynolds flow and as a result we neglect any inertial effect, effectively setting the LHS of (8) and (9) to zero. The result is a local force balance given by

$$(10) \qquad \mathbf{0} = \phi\rho_m\mathbf{g} - \mathbf{f}_I + \nabla\cdot(\phi\boldsymbol{\sigma}_m),$$

$$(11) \qquad \mathbf{0} = (1-\phi)\rho_s\mathbf{g} + \mathbf{f}_I + \nabla\cdot((1-\phi)\boldsymbol{\sigma}_s).$$

Next we need to specify the constitutive relation between $\mathbf{f}_I, \sigma_{m,s}$ and the unknowns $\mathbf{v}_{m,s}, \rho_{m,s}$ to close the system of equations. First we note that the interaction force $\mathbf{f}_I$ has to be independent of the inertial reference frame. Drew & Segel [5] derived a list of functions depending on $\mathbf{v}_{m,s}$

which satisfy this constraint, of which the simplest one is a linear relation to the relative velocity of melt and matrix $\mathbf{v}_m - \mathbf{v}_s$. Furthermore they argued that an interphasial force should also depend on pressure forces in the magma generated by the gradient of the voidage ϕ. Taking the simplest model incorporating these requirements McKenzie arrived at

$$(12) \qquad\qquad \mathbf{f}_I = C(\mathbf{v}_m - \mathbf{v}_s) - P_m\nabla\phi,$$

where C is a for now undetermined constant.

For the stress tensor we distinguish between the magma and the solid matrix. The magma is modelled as an isotropic incompressible fluid which yields

$$(13) \qquad\qquad \boldsymbol{\sigma}_m = -P_m\mathbf{I} + \mu(\nabla\mathbf{v}_m + (\nabla\mathbf{v}_m)^T),$$

where $\mathbf{I}$ is the identity tensor and μ the dynamic viscosity. By averaging over a volume element containing disordered melt channels it is assumed that the strain rate term on the RHS of (14) vanishes by self-cancellation [9] (note that McKenzie does not provide a justification for the omission of the strain)

$$(14) \qquad\qquad \boldsymbol{\sigma}_m = -P_m\mathbf{I}.$$

The stress tensor for the matrix is more involved as we model the solid as a compressible highly viscous fluid in order to account for compaction and deformation of the matrix. This yields the general stress tensor

$$(15) \qquad\qquad \boldsymbol{\sigma}_s = -P_m\mathbf{I} + \eta(\nabla\mathbf{v}_s + (\nabla\mathbf{v}_s)^T) + \left(\zeta - \frac{2}{3}\eta\right)(\nabla\cdot\mathbf{v}_s)\mathbf{I},$$

where η is the matrix shear or dynamic viscosity and ζ is the bulk viscosity. At first sight it might be remarkable to have P_m appearing in the stress tensor for the solid matrix. A more compelling justification is given by Scott & Stevenson [12]. They argue that compaction of the matrix resulting in the melt being expelled is a direct consequence of a pressure difference between melt and matrix, which can be written as

$$(16) \qquad\qquad P_m - P_s = \tilde{\zeta}\nabla\cdot\mathbf{v}_s,$$

where $\tilde{\zeta}$ is an effective matrix bulk viscosity. The pressure is commonly related to the mean normal stress or trace of the stress tensor by

$$(17) \qquad\qquad P_s = -\frac{1}{3}\operatorname{Tr}(\boldsymbol{\sigma}_s).$$

It can then be checked that the choice of P_m in (15) gives the desired compaction-pressure relation (16).

Combining the derived constitutive relations and substituting them in (10) we find

$$(18) \qquad\qquad \mathbf{v}_m - \mathbf{v}_s = -\frac{\phi}{C}(\nabla P_m + \rho_m\mathbf{g}).$$

Using this form, we can find an expression for C by noting that in the case of a steady matrix, i.e. $\mathbf{v}_s = \mathbf{0}$, we would have to retrieve Darcy's law for flow through a porous medium,

$$(19) \qquad\qquad \phi\mathbf{v}_m = -\frac{K}{\mu}(\nabla P_m + \rho_m\mathbf{g}),$$

where K depicts the permeability of the matrix and μ the fluid viscosity. We thus find that we can identify $C = \mu\phi^2/K$ so as to get the right limiting behaviour of the equations

$$(20) \qquad\qquad \phi(\mathbf{v}_m - \mathbf{v}_s) = -\frac{K(\phi)}{\mu}(\nabla P_m + \rho_m\mathbf{g}).$$

Note that the fluid viscosity μ is taken, in this paper, to be constant, although it could depend on variables such as temperature in a more complex model [9]. The permeability, however, is

typically a (non-linear) function of the voidage, i.e. $K(\phi)$. A commonly used relation put forward by Scott & Stevenson [12] specifies a simple power law behaviour

$$(21) \qquad K(\phi) = \frac{a^2\phi^n}{b} = K_0\phi^n,$$

where a is the earlier defined grain size and b a dimensionless parameter, lumped together in K_0.

Substitution of (15) and (10) into (9) and working through the algebra gives

$$(22) \qquad \nabla P_m + \rho_m\mathbf{g} = (1-\phi)\Delta\rho\mathbf{g} + \nabla\cdot\left(\eta^*(\nabla\mathbf{v}_s + (\nabla\mathbf{v}_s)^T)\right) + \nabla\left(\left(\zeta^* - \frac{2}{3}\eta^*\right)\nabla\cdot\mathbf{v}_s\right).$$

Here we have introduced

$$(23) \qquad \Delta\rho = \rho_s - \rho_m,$$

$$(24) \qquad \eta^* = (1-\phi)\eta,$$

$$(25) \qquad \zeta^* = (1-\phi)\zeta,$$

so $\Delta\rho$ is the difference in density between melt and matrix. We therefore see that the melt pressure which drives the porous flow is generated by gravitational buoyancy forces and matrix deformations. Note that both η^* and ζ^* could depend on the porosity. Indeed for $\phi \ll 1$ Scott & Stevenson [12] propose a variation as ϕ^{-m} for $0 \le m \le 1$, whereas the original model by McKenzie [10] assumes constant values or $m = 0$

$$(26) \qquad \eta^* = \eta_0\phi^{-m}, \quad \zeta^* = \zeta_0\phi^{-m}.$$

Now equations (8),(9),(20),(22) together with constitutive relations for $\eta^*, \zeta^*, K, \mu, \Gamma$ form the full set of McKenzie equations. These can be solved for unknowns $\mathbf{v}_{m,s}, P_{m,s}$ if we prescribe $\rho_{m,s}$-equations or for $\mathbf{v}_{m,s}, \rho_{m,s}$ if we prescribe $P_{m,s}$-equations.

2.2. **Boussinesq approximation.** Progress on this set of equations turns out to be complicated and therefore we first simplify the system slightly by using a Boussinesq approximation. We assume that the densities of the melt and matrix are constants (not necessarily equal) and that the difference is negligibly small, except where it is of influence in gravitational terms where we assume it is a constant. This reflects the idea that buoyancy is a crucial factor in driving the melt flow through the matrix. Next we write $\mathbf{g} = g\hat{\mathbf{g}}$, where $\hat{\mathbf{g}}$ is the unit vector in the direction of gravitational pull. Therefore we can also redefine the fluid pressure by incorporating lithostatic pressure directly into the fluid pressure

$$(27) \qquad P = P_m - \rho_m g\mathcal{G},$$

where $\nabla\mathcal{G} = -\hat{\mathbf{g}}$ (in case of $\hat{\mathbf{g}} = -\hat{\mathbf{k}}$, the vertical downward direction, we have $\mathcal{G} = z$). This yields a new similar set of equations which we will consider from now on

$$(28a) \qquad \frac{\partial\phi}{\partial t} + \nabla\cdot(\phi\mathbf{v}_m) = \frac{\Gamma}{\rho_m},$$

$$(28b) \qquad -\frac{\partial\phi}{\partial t} + \nabla\cdot((1-\phi)\mathbf{v}_s) = -\frac{\Gamma}{\rho_s},$$

$$(28c) \qquad -\frac{K(\phi)}{\mu}\nabla P = \phi(\mathbf{v}_m - \mathbf{v}_s),$$

$$(28d) \qquad \nabla P = (1-\phi)\Delta\rho g\hat{\mathbf{g}} + \nabla\cdot\left(\eta^*(\nabla\mathbf{v}_s + (\nabla\mathbf{v}_s)^T)\right) + \nabla\left(\left(\zeta^* - \frac{2}{3}\eta^*\right)\nabla\cdot\mathbf{v}_s\right),$$

where $\Delta\rho = \rho_s - \rho_m$ is the difference in density between melt and matrix and is assumed to be constant. The first two equations can be summed to yield

$$(29) \qquad \nabla\cdot(\phi\mathbf{v}_m + (1-\phi)\mathbf{v}_s) = \frac{\Delta\rho\Gamma}{\rho_m\rho_s}.$$

Note that using actual estimates of the densities $\rho_m \approx 2.8 \cdot 10^3$ kg m^{-3} and $\rho_s \approx 3.3 \cdot 10^3$ kg m^{-3} from McKenzie [10] we find that $\Delta\rho/\rho_m\rho_s \sim 10^{-4}$ which justifies neglecting the term containing the melting Γ in many cases (termed the extended Boussinesq approximation by Katz [9]).

2.3. **Non-dimensionalisation.** In order to further analyse the set of equations they need to be non-dimensionalised. In literature it appears to be customary to first use further approximations to simplify the equations, e.g. [3, 11]. It is however possible to pursue the process of non-dimensionalising on the above set of equations derived by the Boussinesq approximation trying to motivate as many of the rescaling on physical arguments.

An important feature is the determination of the so-called compaction length δ as given by McKenzie in his original paper, but lacking a clear motivation. We will firstly set

$$(30) \qquad \mathbf{x} = \delta\hat{\mathbf{x}}, \quad t = T\hat{t}, \quad \mathbf{v}_{m,s} = V\hat{\mathbf{v}}_{m,s}, \quad \Gamma = \Gamma_0\hat{\Gamma}, \quad \rho_{m,s} = \rho_0\hat{\rho}_{m,s}.$$

By using (6),(7) we find that, as might be expected, the time scales as $T = \delta/V$ and melting rate as $\Gamma_0 = \rho_0/T$.

In order to use the momentum equations we need to find a suitable scale for the pressure. Motivated by the buoyancy driven pressure gradient to migrate melt we choose to scale pressure on a buoyancy scale. The scale of the matrix viscosities η, ζ will be set by the same parameter as proposed by Scott & Stevenson [12]

$$(31) \qquad P = \Delta\rho g\delta\hat{P}, \quad K = K_0\hat{K}, \quad \eta^* = \xi_0\hat{\eta}, \quad \zeta^* = \xi_0\hat{\zeta}.$$

The conservation of magma momentum (28c) leads to $V = K_0\Delta\rho g/\mu$. Using this information in the conservation of matrix momentum (28d) then yields $\delta^2 = \xi K_0/\mu$. McKenzie proposes to scale the viscosities, based on the equation for constant η, ζ, with $\xi = \zeta_0 + 4\eta_0/3$ (see appendix A.1). In doing so we retrieve the compaction length proposed by McKenzie [10]

$$(32) \qquad \delta_c = \sqrt{\frac{(\zeta_0 + 4\eta_0/3)K_0}{\mu}}, \quad T = \frac{1}{\Delta\rho g}\sqrt{\frac{\mu(\zeta_0 + 4\eta_0/3)}{K_0}},$$

which is the natural length scale for the reaction of the matrix-melt to a compaction (see appendix A.2).

Now we find upon using the rescaling and by dropping of the hats the non-dimensionalised Boussinesq approximation of the McKenzie model

$$(33a) \qquad \frac{\partial\phi}{\partial t} + \nabla \cdot (\phi\mathbf{v}_m) = \frac{\Gamma}{\rho_m},$$

$$(33b) \qquad -\frac{\partial\phi}{\partial t} + \nabla \cdot ((1-\phi)\mathbf{v}_s) = -\frac{\Gamma}{\rho_s},$$

$$(33c) \qquad -K\nabla P = \phi(\mathbf{v}_m - \mathbf{v}_s),$$

$$(33d) \qquad \nabla P = (1-\phi)\hat{\mathbf{g}} + \nabla \cdot \left(\eta(\nabla\mathbf{v}_s + (\nabla\mathbf{v}_s)^T)\right) + \nabla\left(\left(\zeta - \frac{2}{3}\eta\right)\nabla \cdot \mathbf{v}_s\right).$$

Boundary conditions need to be prescribed to make the problem well-posed. These depend on the specific problem set-up and will be posed at the subsequent places in the paper.

3. SOLITARY WAVES

No direct solutions are know to date to the full McKenzie model and it is mainly studied using numerics. Slight simplifications of the Boussinesq approximation of the McKenzie model however offer rich behaviour as we will see.

We will from now on direct our attention to a one-dimensional column of matrix-melt with vertical coordinate z where gravity points in the negative z-direction. All the derivatives in space thus become partial derivatives in z. We also assume that there is no active melting

process transferring matrix into magma or vice versa, i.e. $\Gamma = 0$. We choose (21) and (26) as specific models for permeability and viscosity. Two very similar sets of equations have been derived on these assumptions, by Scott & Stevenson [13], which in literature is often termed the magma equation, and by Barcilon & Richter [3].

Magma equation. To derive the magma equation we assume a low porosity matrix-melt system, i.e. $\phi \ll 1$. This results in $(1 - \phi) \approx 1$ which gives upon elimination of P and v_m

$$\text{(34a)} \qquad \frac{\partial \phi}{\partial t} = \frac{\partial v_s}{\partial z},$$

$$\text{(34b)} \qquad v_s = \phi^n \left(-1 + \frac{\partial}{\partial z} \left(\phi^{-m} \frac{\partial v_s}{\partial z} \right) \right),$$

or written as a single non-linear PDE in ϕ, the magma equation

$$\text{(35)} \qquad \frac{\partial \phi}{\partial t} = \frac{\partial}{\partial z} \left[\phi^n \left(-1 + \frac{\partial}{\partial z} \left(\phi^{-m} \frac{\partial \phi}{\partial t} \right) \right) \right].$$

Barcilon & Richter equation. If we instead make an extra rescaling on the assumption that we are given a reference state ϕ_0 without assuming it being small, we arrive at a different equation. The assumption of the functional form of the viscosities η and ζ was suggested to be valid for $\phi \ll 1$ and we therefore do not assume this for this derivation. Instead we pick them both equal and constant. We rescale now with $\phi = \phi_0 \hat{\phi}$ and as ϕ is already a dimensionless number we can choose freely where to incorporate this new quantity as long as it is done so consistently. Following Barcilon & Richter we choose $\tilde{t} = \frac{\phi_0}{1 - \phi_0} t$ and $\tilde{v}_{m,s} = (1 - \phi_0) v_{m,s}$. Furthermore we choose $K_0 = K(\phi_0)$. After elimination of P and v_m and dropping of the tildes this yields

$$\text{(36a)} \qquad \frac{\partial \phi}{\partial t} = \frac{\partial}{\partial z} \left((1 - \phi_0 \phi) v_s \right),$$

$$\text{(36b)} \qquad \frac{v_s}{\phi^n} = -\frac{1 - \phi_0 \phi}{1 - \phi_0} + \frac{\partial^2 v_s}{\partial z^2}.$$

This shows the motivation for the rescaling, as now the state $\phi = 1$, $v_s = -1$ provides a steady state of the system, a uniform compacting solid matrix. Note that for $\phi_0 \ll 1$ we recover the magma equation (35) for $m = 0$.

3.1. **Solitary wave solutions.** Scott & Stevenson were one of the first to look for solutions of (35) using numerics [13]. They considered, among other things, the set-up where a region with high magma fraction lies beneath a region with low magma fraction. In this case they observed the formation of what appeared to be solitons, strongly localised travelling waves which retain their shape while moving. The soliton-like solutions appeared to interact cleanly, retaining their shape and leaving the background porosity undisturbed. This led them to believe that the solutions were in fact solitons and thus they proposed the name magmons to describe this type of waves.

Analytical progress can, in fact, be made as was showed by Scott & Stevenson in the same paper. They derived the wavespeed for the travelling waves for the magma equation in the cases $m = 0$ and $m = 1$. A major drawback of the magma equation in applicability is, however, that it is only valid for small ϕ thereby possibly neglecting higher order effects. In spirit of Barcilon & Richter, will therefore consider the case $m = 0$, i.e. constant matrix viscosities, more rigorous by considering (36a),(36b) for general background voidage ϕ_0.

Travelling wave transformation. We look for travelling wave solutions by switching to a travelling wave coordinate $y = z - ct$, where c is the wavespeed. We assume the solutions to be of the form

$$\phi(z,t) = f(z - ct) = f(y), \tag{37a}$$

$$v_s(z,t) = g(z - ct) = g(y). \tag{37b}$$

We look for a specific type of travelling wave as suggested by observations, a solitary wave. Therefore we assume that f and g strongly decay to the background state at infinity,

$$f \to 1, f' \to 0, f'' \to 0 \qquad\qquad \text{as } |y| \to \infty, \tag{38a}$$

$$g \to -1, g' \to 0, g'' \to 0 \qquad\qquad \text{as } |y| \to \infty. \tag{38b}$$

Here differentiation with respect to y is depicted by a prime. The travelling wave transformation changes the PDEs (36a) and (36b) into a set of two ODEs

$$-cf' = ((1 - \phi_0 f)g)', \tag{39a}$$

$$\frac{g}{f^n} = -\frac{1 - \phi_0 f}{1 - \phi_0} + g''. \tag{39b}$$

Equation (39a) can be integrated directly. By employing the far-field conditions to explicitly determine the integration constant, we can then express g as

$$g = -\frac{c(f - 1) + (1 - \phi_0)}{1 - \phi_0 f}. \tag{40}$$

Next we set $p = f'$ and note the identity $dp^2/df = 2f''$. This and (40) allow us to reformulate (39b) as an ODE for p

$$\frac{c(f - 1) + (1 - \phi_0)}{f^n(1 - \phi_0 f)} = \frac{1 - \phi_0 f}{1 - \phi_0} + \frac{1}{2}(1 - \phi_0)(c + \phi_0)(1 - \phi_0 f)^2 \frac{d}{df}\left(\frac{p^2}{(1 - \phi_0 f)^4}\right). \tag{41}$$

An interesting consequence of this reformulation is that we can interpret the solitary waves as homoclinic orbits in (p, f) phase space governed by (41). The orbits must emanate from the point $(0, 1)$ and make an excursion through the positive f-half-plane. The orbit must ultimately return in a symmetric fashion to $(0, 1)$. The maximal amplitude A of the solitary wave is, consequently, found to be the crossing of the f-axis, i.e. when $p = 0$.

To calculate the trajectories, and thus the waveform, and wavespeed of the solitary waves we have to work through some cumbersome algebra. We will only sketch the general procedure here and refer the interested reader for details to appendix A.3.

Amplitude dispersion. Firstly we will derive the phase speed c of the solitary waves. We write (41) in a more condensed form

$$\lambda \frac{d}{df}\left(\frac{p^2}{(1 - \phi_0 f)^4}\right) = ch_1(f, n, \phi_0) + h_2(f, n, \phi_0), \tag{42}$$

for some functions h_1, h_2 and constant λ. By formal integration over the trajectory, i.e. starting from $p = 0, f = 1$, with respect to f from 1 to x we thus find that

$$p^2 = \frac{(1 - \phi_0 x)^4}{\lambda}\left(cH_1(x, n, \phi_0) + H_2(x, n, \phi_0)\right), \tag{43}$$

where $H_1(x), H_2(x)$ are the definite integrals of h_1, h_2 from 1 to x.

Then we recall that at the point of maximal amplitude A we have $p = 0$ and thus find that

$$0 = \frac{(1 - \phi_0 A)^4}{\lambda}(cH_1(A, n, \phi_0) + H_2(A, n, \phi_0)). \tag{44}$$

As a result we find a wavespeed c which depends on the maximal amplitude A,

$$(45) \qquad c = \mathcal{C}(A, n, \phi_0) = -\frac{H_2(A, n, \phi_0)}{H_1(A, n, \phi_0)},$$

for the more detailed form of c, see appendix A.3. Figure 2 shows the phase speed $\mathcal{C}(A, n, \phi_0)$ as a function of the amplitude A for various n and ϕ_0.
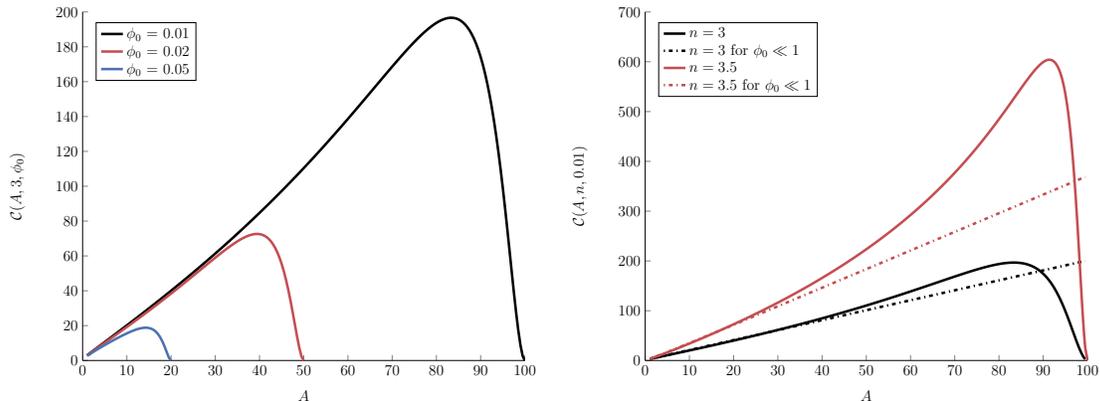
(A) Phase speed as a function of the maximal amplitude A of the wave for $n = 3$ and different background voidage levels ϕ_0.

(B) Phase speed as a function of the maximal amplitude A of the wave for different n, background voidage $\phi_0 = 0.01$. A comparison with the approximation found by considering $\phi_0 \ll 1$ is plotted as dashed lines.

FIGURE 2. Behaviour of the phase speed $c = \mathcal{C}(A, n, \phi_0)$ as a function of its arguments. Figure 2a shows the (non-linear) dependency on ϕ_0. Figure 2b depicts the dependency on n and shows comparison with low voidage linearisation results, as found for example in [13].

As noted from the expression for c and its plot, the wavespeed depends (strongly) on the amplitude of the wave. This amplitude dispersion is a general feature for non-linear waves and can be observed in many non-linear wave models, such as the KdV equation. We can infer from the dispersion relation that if multiple solitary waves with different amplitude A are to co-exist they will have to interact with one another. The waves have different speeds and resultantly cross each other, initiating a strong interaction between the waves.

As an interesting side note we recall that the (36a) and (36b) in the limit of $\phi_0 \ll 1$ will converge towards (35). Scott & Stevenson derived a wavespeed based on this model and this should thus agree, at least to linear order, with our results. It is thus reassuring to observe that both agree up to order ϕ_0 for $A \to 1$, see figure 2b and appendix A.3.

Waveform. From (43) we can infer p and thus df/dy. This allows us to find an implicit description of the waveform, $f(y)$, by using

$$(46) \qquad y(x) = \int_x^A \frac{dy}{df}\,\mathrm{d}f = \int_x^A \frac{1}{p(f)}\,\mathrm{d}f.$$

Therefore, given A, ϕ_0 and n, we are in principle able to calculate $y(f)$ which gives upon inversion $f(y)$ the shape of the wave. In turns out that in practice (46) is infeasible due to the complicated expression for $p(f)$. The integral can of course be approximated using numerical integration. This yields the waveforms depicted in figure 3.

We see that the travelling waves are indeed strongly decaying towards infinity, making them localised pulses and thus solitary waves. The effect of the amplitude, apart from drastic effects on the wave speed, seems to be merely a scaling of the waveform, see figure 3a. The background porosity ϕ_0 appears to control the localisation of the solitary wave, see figure 3b. We thus expect to see sharply localised pulses for low background voidage.
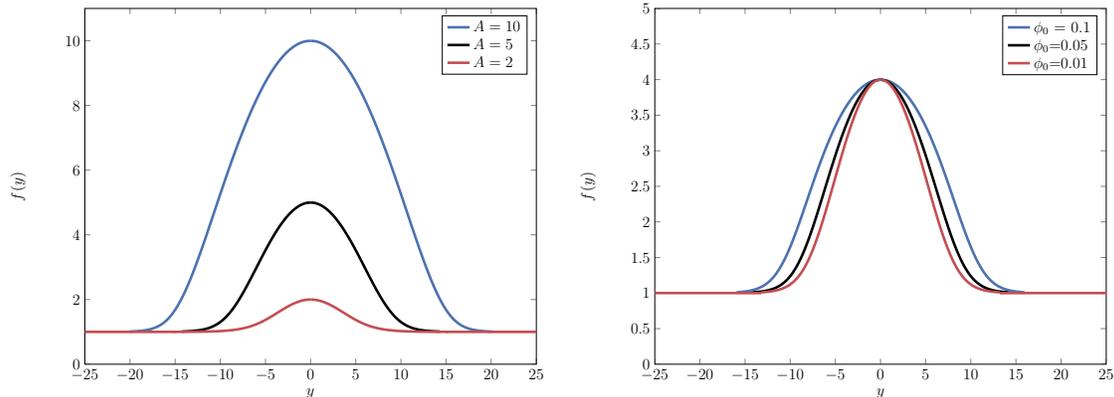
(A) Waveform $f(y)$ as a function of travelling wave coordinate $y = z - ct$ for different amplitudes A. The background voidage is $\phi_0 = 0.01$ in all cases and $n = 3$.

(B) Waveform $f(y)$ as a function of travelling wave coordinate $y = z - ct$ for different background voidage ϕ_0. The amplitude $A = 4$ in all cases and $n = 3$.

FIGURE 3. Waveform of the solitary wave for different parameters. For varying amplitude A, see figure 3a. For varying background voidage ϕ_0, see figure 3b. The background voidage seems to control the width of the solitary wave, making it more localised for smaller values of background voidage.

Note that the width of the solitary waves is of the order of 10 in non-dimensional units which rescales to the same amount of compaction lengths. Using estimates of the compaction length, on the order of 10 km [12], we conclude that magmons should be measurable, if they are a real phenomenon. The wavespeed in dimensional units is estimated to be of the order of 10 cm per year, which should be observable as well.

3.2. **Solitons or solitary waves?** Barcilon & Richter noted that in numerical simulations of (36a),(36b) where two unequal sized solitary waves interact, an imperfection is observed [3]. A dispersive tail remains after collision of the waves, disrupting the background porosity. Consequently, an interesting question one may ask upon this observation is, whether the travelling wave solutions in the preceding models are pure solitons or are merely solitary waves. As defined by Drazin & Johnson a soliton is *"any solution of a nonlinear evolution equation (or system) which (i) represents a wave of permanent form; (ii) is localised, so that it decays or approaches a constant at infinity; (iii) can interact strongly with other solitions and retain its identity"* [4].

Equations admitting soliton solutions are rather exceptional. They have to admit a solution using the inverse scattering transform (IST), which puts a severe restriction on the type of equations, these equations are namely said to be completely integrable. There is an intimate link between the notion of integrability and conservation laws of the PDE as it is conjectured that completely integrable systems possess an infinite set of these conserved quantities. This provides a way to strongly suggest non-existence of soliton solutions for a PDE.

Another conjecture relates the complete integrability of a system to ODEs that can be derived from it by exact reduction. It is conjectured that all of these differential equations have to satisfy

the Painlevé property. This again serves as a way to suggest non-existence of solitons. It has to be noted, however, that as of yet these methods do not provide concluding evidence as the conjectures remain unproven.

Conservation laws for PDEs. In looking for a conservation law for an evolution equation in variables t, z one looks for T, X such that

$$(47) \qquad \frac{\partial T}{\partial t} = \frac{\partial X}{\partial z},$$

so that we find a constant of motion by noting that

$$(48) \qquad \frac{\partial}{\partial t} \int_\Omega T \, \mathrm{d}z = 0.$$

Functions of the form $T = f_z$, $X = f_t$ are excluded because they trivially satisfy (47). As one needs to be able to evaluate T at $t = 0$ its dependence on its arguments is restricted to functions which can be evaluated from the initial conditions.

Strengthened by their observation of dispersive interaction between two solitary wave type solutions Barcilon & Richter set out to find conservation laws for (36a),(36b) for $n = 3$ and in the approximation of low background voidage ϕ_0 [3], thus in effect (35) for $n = 3$ and $m = 0$. Given the constraint of dependency on the initial data we look for conservation laws of the form

$$(49a) \qquad T = T_n(\phi, \phi_z, \ldots, \phi_{nz}),$$
$$(49b) \qquad X = X_n(\phi, \phi_z, \ldots, \phi_{nz}, \phi_t, \phi_{tz}),$$

where ϕ_{nz} represents the n-th derivative with respect to z. Note that higher time derivatives can be expressed in space derivatives and ϕ_t, ϕ_{tz} by use of the PDE and thus need not be included in the arguments of T, X explicitly. Barcilon & Richter were able to only find two such conserved quantities T_0, X_0 and T_1, X_1, while proving that no conservation law for T_3, X_3 is possible. This suggests that the observed solutions are indeed not solitons.

A more general and rigorous treatment was performed by Harris [7], who considered the magma equation for all constant values m, n. She concluded that in general only two conservation laws exist. The first one given by

$$(50a) \qquad T_1 = \phi - 1,$$
$$(50b) \qquad X_1 = m\phi^{n-m-1}\phi_t\phi_z - \phi^{n-m}\phi_{tz} + \phi^n,$$

and has as interpretation that it is linked to conservation of matrix mass. Another conservation law of first order is found which depends on the specific choice of m, n, but is given for $m \neq 1, 1-n, 2-n$ by

$$(51a) \qquad \tilde{T}_1 = \frac{1-m-n}{2}\phi^{-2m}\phi_z^2 + \frac{1}{2-m-n}\left(\phi^{2-m-n} - 1\right)$$
$$(51b) \qquad \tilde{X}_1 = m\phi^{-2m}\phi_t\phi_z - \phi^{1-2m}\phi_{tz} + \frac{n}{1-m}\phi^{1-m}$$

generalising the results found by Barcilon & Richter. In addition Harris showed that higher order conservation laws do not exist unless m, n take the combination $m = n+1, n \neq 0$ or $m = 1, n \neq 0$ confirming the hypothesis by Barcilon & Richter that the equations only have a finite number of conserved quantities.

Note that the case $m = 1, n \neq 0$ is the only of the two remaining combinations which could have physical meaning as $0 \leq m \leq 1, 2 \leq n \leq 3$ is expected from the modelling of viscosity and permeability.

Painlevé property. PDEs can be reduced to ODEs by certain types of transformation, for example by changing to a travelling wave frame or by similarity reduction. The type of ODE coming from these reductions and integrability of the evolution equation appear to be intimately linked due the following conjecture

Conjecture 1 (Ablowitz-Ramani-Segur [1]). Consider an evolution equation in the IST class. Then every ODE obtained by an exact reduction of the evolution equation has the Painlevé property.

The Painlevé property concerns second order ODEs of the form

$$(52) \qquad \frac{d^2 w}{dz^2} = F\left(w, \frac{dw}{dz}, z\right),$$

where F is a rational function in w and dw/dz. If these ODEs do not posses a movable singularity in their solution they are said to posses the Painlevé property. The name honours Painlevé, who showed that only a restricted class of 50 equations satisfy this property.

As noted by Drazin & Johnson a proof (or disproof) of the conjecture is still unknown, but strong evidence points towards it being true. Many of the equations with soliton solutions by the IST, such as the KdV equation and the non-linear Schrödinger equation, have indeed showed to confirm the conjecture.

Using the reduction by travelling wave coordinates Harris [8] was able to show that for integer m, n only $m = 0, n = -1$ and $m = 0, n = -2$ yield an ODE satisfying the Painlevé property[1]. Combining this result with the lack of conservation laws found by Harris earlier strengthens the hypothesis that for the physical acceptable magma models the solutions are not of a soliton type and thus are likely to be mere solitary waves.

4. Conclusion

In this paper we derived a model for the flow of magma through the earths mantle by using a two-phase flow. We describe the mantle as a solid permeable matrix through which the viscous magma fluid gets transported.

The model is simplified using a Boussinesq approximation and assuming special forms for the constitutive laws governing permeability and viscosity of the matrix. This yields a set of equations first described by Barcilon & Richter [3]. Analytical expressions are found describing solitary waves in this model confirming the numerical experiments performed by various authors [3, 13].

Based on numerical simulations of the interaction between such waves, it was hypothesised that the waves are not pure soliton solutions [3]. There exists strong evidence confirming this hypothesis. Firstly, only a finite number of conservation laws can be found for the evolution equation. Secondly, the reduction from the evolution equation to an ODE by a travelling transformation results in an ODE which does not posses the Painlevé property. Both properties are conjectured to be necessary conditions for an evolution equation to admit soliton solutions.

A possible extension of the work carried out in this paper is the obvious step from one to higher-dimensional formulation of the equations. An immediate question arises in this context, are the solitary waves found stable in more dimensions? It has been proved in the literature that this is not the case [2] and that the solitary waves break up to form circular localised waves in two dimensions [12].

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[1]By further analysis using similarity solution reduction it can be shown that the list of admissible rational m, n can be reduced further to $m = 0, n = -1$ and $m = 1/2, n = -1/2$. In both cases the magma equation can be transformed into PDE which is known to be completely integrable [8].

Appendix A. Appendix

A.1. **Matrix viscosity rescaling.** Motivation for the η, ζ rescaling comes from considering the constant viscosities problem. In that case (28d) can be written as

$$(53) \qquad \nabla P = (1 - \phi)\Delta\rho g\hat{\mathbf{g}} - \eta^*\nabla \times \nabla \times \mathbf{v}_s + \left(\zeta^* + \frac{4}{3}\eta^*\right)\nabla(\nabla \cdot \mathbf{v}_s).$$

Here we have used the vector identities $\nabla \cdot \nabla\mathbf{v} = \nabla(\nabla \cdot \mathbf{v})$, $\nabla \cdot (\nabla\mathbf{v})^T = \nabla^2\mathbf{v}$ and $\nabla^2\mathbf{v} = \nabla(\nabla \cdot \mathbf{v}) - \nabla \times \nabla \times \mathbf{v}$. In an irrotational matrix flow the curl term vanishes and it can be observed that $\zeta^* + 4\eta^*/3$ is the natural occurring matrix viscosity. Choosing the rescaling as proposed by McKenzie the equation thus takes a very simple form in the irrotational limit

$$(54) \qquad \nabla P = (1 - \phi)\hat{\mathbf{g}} + \nabla^2\mathbf{v}_s.$$

Note that this rescaling can also be achieved by considering the problem in one dimension, as in this case the curl term is absent as well. The result is the same rescaled equation with $\nabla = \frac{d}{dz}$.

A.2. **Compaction length.** A basic interpretation of the compaction length was offered by McKenzie [10]. Consider the problem of a one-dimensional initial constant porosity mantle column being compressed by an impermeable horizontal surface moving with speed V. We take $z \in [0, \infty)$ for the domain. The no-slip boundary condition comes from the impermeability of the compression surface, i.e. $v_{m,s}(0) = V$. For the pressure, we assume that for $z \to \infty$ the pressure is undisturbed and thus $P \to 0$. We consider this problem only on a small time-scale such that $\phi \approx 1$, the reference porosity. By neglecting buoyancy terms and melting, the equations become

$$(55) \qquad \frac{d}{dz}(v_m - v_s) = -\frac{1}{\phi_0}\frac{dv_s}{dz},$$

$$(56) \qquad -\frac{dP}{dz} = (v_m - v_s),$$

$$(57) \qquad \phi_0\frac{dP}{dz} = \frac{d^2v_s}{dz^2}.$$

We can combine these into

$$(58) \qquad \frac{d^3v_s}{dz^3} = \frac{dv_s}{dz}.$$

Using the boundary condition we thus find that bounded solutions satisfy

$$(59) \qquad v_s(z) = Ve^{-z}, \quad v_m(z) = V + \frac{V(1 - \phi_0)}{\phi_0}(e^{-z} - 1), \quad P(z) = -\frac{e^{-z}}{\phi_0}.$$

The compaction rate of the matrix is defined by

$$(60) \qquad \frac{dv_s}{dz} = -Ve^{-z}.$$

Recall that all these expressions are in non-dimensional units and thus that the effect of the compression at the surface of the column has a decay factor of δ in dimensional units. The compaction length is thus the length over which the compaction rate dies out by a factor $1/e$, hence the name compaction length.

A.3. **Derivation wave speed and waveform.** To derive a more detailed expression for the wave speed c we return to (41) and the related equation (42). We can make the following identification of λ, h_1 and h_2

$$(61a) \qquad\qquad \lambda = \frac{1}{2}(1 - \phi_0)(c + \phi_0),$$

$$(61b) \qquad\qquad h_1(f, n, \phi_0) = \frac{1 - f}{f^n(1 - \phi_0 f)^3},$$

$$(61c) \qquad\qquad h_2(f, n, \phi_0) = \frac{1}{(1 - \phi_0)(1 - \phi_0 f)} + \frac{\phi_0 - 1}{f^n(1 - \phi_0 f)^3}.$$

Integrating h_1 and h_2 with respect to the amplitude from 1 to f then yields

$$(62a) \qquad\qquad H_1(f, n, \phi_0) = \int_1^f \frac{1 - x}{x^n(1 - \phi_0 x)^3} \, \mathrm{d}x,$$

$$(62b) \qquad\qquad H_2(f, n, \phi_0) = \frac{-1}{\phi_0(1 - \phi_0)} \log\left(\frac{1 - \phi_0 f}{1 - \phi_0}\right) + \int_1^f \frac{\phi_0 - 1}{x^n(1 - \phi_0 x)^3} \, \mathrm{d}x.$$

The resulting wave speed is given by the very appealing formula

$$(63) \qquad c = \mathcal{C}(A, n, \phi_0) = -\frac{\dfrac{1}{\phi_0(1 - \phi_0)} \log\left(\dfrac{1 - \phi_0 A}{1 - \phi_0}\right) + \displaystyle\int_1^A \frac{1 - \phi_0}{x^n(1 - \phi_0 x)^3} \, \mathrm{d}x}{\displaystyle\int_1^A \frac{x - 1}{x^n(1 - \phi_0 x)^3} \, \mathrm{d}x}.$$

This closely resembles the formula given in [3], although it seems there are some typos in their derivation.

The process of calculating the shape is very tedious. We start out by expanding (43) to get

$$(64) \qquad p^2(f, n, \phi_0) = \frac{2(1 - \phi_0 f)^4 \left(\mathcal{C}(A, n, \phi_0) - \mathcal{C}(f, n, \phi_0)\right)}{(1 - \phi_0)(\mathcal{C}(A, n, \phi_0) + \phi_0)} \int_1^f \frac{1 - x}{x^n(1 - \phi_0 x)^3} \, \mathrm{d}x,$$

where we now have used that $c = \mathcal{C}(A, n, \phi_0)$ for the maximal amplitude A of the wave. By following the procedure described in the main body one can now calculate an implicit expression for the waveform

$$(65) \qquad\qquad y(f) = \int_f^A \frac{1}{p(x, n, \phi_0)} \, \mathrm{d}x.$$

As pointed out by Barcilon & Richter the calculation of the integral can be difficult using numerical techniques as we know that $p(A) = 0$ and thus (65) shows a diverging integrand at one of the endpoints (note that this does not mean that the integral is not well-defined). One way to numerically elude this issue is to split the integral

$$(66) \qquad\qquad y(f) = \int_f^{A - \epsilon} \frac{1}{p(x, n, \phi_0)} \, \mathrm{d}x + \int_{A - \epsilon}^A \frac{1}{p(x, n, \phi_0)} \, \mathrm{d}x.$$

Now note that we can approximate the latter integral by using a Taylor approximation of p^2 around $x = A$. This expansion is given by

$$p^2(x, n, \phi_0) = \left.\frac{\partial p^2}{\partial x}\right|_{x=A} (x - A) + \mathcal{O}\left((x - A)^2\right)$$

$$(67) \qquad\qquad\qquad = 2\kappa(A - x) + \mathcal{O}\left((x - A)^2\right),$$

where $\kappa = -f''(A)$ the negative curvature of the waveform at the maximal amplitude (note that $\kappa > 0$ for a solitary wave). This finally yields

$$(68) \qquad y(f) = \int_f^{A-\epsilon} \frac{1}{p(x,n,\phi_0)}\,dx + \sqrt{\frac{2\epsilon}{\kappa}},$$

which is numerically more stable.

Comparison with Scott & Stevenson[13]. Upon taking $\phi_0 \to 0$ in (41) we recover the travelling wave formulation for the magma equation as derived by Scott & Stevenson [13]

$$(69) \qquad \frac{1}{2}c\frac{dp^2}{df} = \frac{c(f-1)+1}{f^n} - 1.$$

Following the same procedure as described in this paper (but much less cumbersome!) one can now derive an expression for c and this yields the result by Scott & Stevenson

$$(70a) \qquad \begin{cases} c(A) = \dfrac{(A-1)^2}{A\log(A) - A + 1}, & \text{for } n = 2, \\[2ex] c(A) = (n-2)(n-1)\dfrac{A + (A^{1-n}-n)/(n-1)}{1+(n-2)A^{1-n}-(n-1)A^{2-n}} & \text{for } n > 2. \end{cases}$$

We would like to point out that $\lim_{A\to 1} c(A) = n$ in this case. Taking the limit of $A \to 1$ for $\mathcal{C}(A,n,\phi_0)$ we obtain

$$(71) \qquad \lim_{A\to 1} \mathcal{C}(A,n,\phi_0) = n - (n+2)\phi_0.$$

Consequently, we find that the difference between them is indeed of linear order in ϕ_0.

Especially noteworthy is the case $n = 3$, as was shown by Barcilon & Richter [3]. In this case the approximation yields a particularly easy form for the phase speed, $c(A) = 2A + 1$. This special case allows us to explicitly integrate (65)

$$y(f) = \pm\sqrt{A + \frac{1}{2}} \int_f^A \frac{x}{(x-1)\sqrt{A-x}}\,dx$$

$$(72) \qquad = \pm\sqrt{A + \frac{1}{2}\left(-2\sqrt{A-f} + \frac{1}{\sqrt{A-1}}\log\left(\frac{\sqrt{A-1}-\sqrt{A-f}}{\sqrt{A-1}+\sqrt{A-f}}\right)\right)},$$

which closely resembles the waveform for small ϕ_0. It looks as if the case $n = 3$ is however of a very special type as the explicit integration cannot be easily extended to other values of n.

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